JOURNAL OF COMPUTATIONAL PHYSICS **78**, 369–377 (1988)

An Efficient Method for Subtracting off Singularities at Corners for Laplace's Equation

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Received August 11, 1986; revised July 21, 1987

1. INTRODUCTION

There are many methods for approximating the solution of a boundary value problem for Laplace's equation, e.g., finite difference or finite element methods and boundary integral methods. None of these methods is accurate, however, when the solution has a singularity on the boundary which is due either to a (geometric) corner or to an abrupt shift in boundary values.

Many methods have been proposed to treat these difficulties, only some of which shall be mentioned here. The use of the asymptotic behavior of the solution is one, and Fix *et al.* [2] included some of the lower order singular functions of the asymptotic expansion in their space of trial functions for the finite element method. Another consideration is that of Li [5] who combines the finite element method, used away from the corner, with a Ritz–Galerkin technique, using the singular functions near the corner. Still another approach is the boundary integral equation method, see, e.g., Xanthis *et al.* [11]. There is also, of course, the method of refining the mesh (finite-difference or finite-element methods are usually involved) in a neighborhood of the corner, see, e.g., Gregory *et al.* [3] or Thatcher [6]. In Whiteman and Papamichael [9] conformal mapping techniques were used, and special isoparametric elements at the corner were used by Wait [7] and Henshell and Shaw [4]. See also Whiteman [8] for the dual-series method.

In this paper a very simple method for subtracting off the singularity will be considered. The method is based on an algorithm (see Wigley [10]) for calculating the coefficients of the asymptotic expansion of the solution at the singularity. The method has the great advantage of allowing existing computer programs to be modified in a relatively easy manner in order to calculate as many coefficients as desired. In particular, the first coefficient, sometimes known as the stress intensity factor, can be computed with accuracy and ease.

These calculated coefficients are then used to "subtract off the singularity" in the sense of modifying the original boundary value problem, solving the modified one, and continuing the process. Some numerical results will be given which (a) test the method by comparing some calculated solutions with exact (known) solutions and

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(b) by comparing some calculated solutions to problems with unknown solutions to results given in some of the above papers.

2. Subtracting Off the Singularity

For the sake of example we first consider the well-known problem of Motz, which has become a benchmark for problems of the sort being considered. In the rectangle with vertices $(\pm 1, 0)$ and $(\pm 1, 1)$ is sought a harmonic function $u(x, y)$ whose outward normal derivative vanishes on the top and the right-hand side of the rectangle, as well as on the segment $y=0$, $x<0$; and for which $u=0$ on $y=0$, $x>0$, and $u=500$ on the left-hand side of the rectangle (see Fig. 1).

As is well known, the solution $u(x, y)$ has a singularity at the origin and is asymptotic to a series of the form

$$\sum_{k=0}^{\infty} a_k r^{(2k+1)/2} \sin((2k+1)\theta/2). \tag{1}$$

We now propose a numerical scheme for calculating the coefficients a_k, based on [10]. Let Γ_δ and Γ_A be two semi-circles in the upper half plane, concentric about the origin, and of radii δ and A, respectively, with $0<\delta<A\leqslant 1$. Let k be a positive integer and let v be defined on the upper half plane by

$$v = (r^{-(2k+1)/2} - A^{-(2k+1)} r^{(2k+1)/2}) \sin((2k+1)\theta/2).$$

Observe that v is harmonic and assumes the same boundary values on the x-axis as does the solution u. In addition, v vanishes on the arc $r=A$ and is singular at the origin. Green's theorem is now applied to the functions u and v over the half-annulus which is bounded by the two above-mentioned semi-circles and the x-axis. Since u and v are both harmonic and satisfy the same homogeneous boundary conditions on the x-axis, it follows directly from Green's theorem that

$$\int_{\Gamma_\delta} (uv_r - vu_r)\, ds = \int_{\Gamma_A} (uv_r - vu_r)\, ds, \tag{2}$$

where both integrals are taken in the counterclockwise sense.

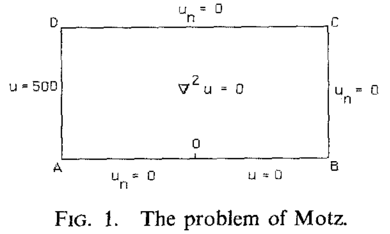

Fig. 1. The problem of Motz.

Let K be a positive integer and let u be replaced by its asymptotic expression

$$\sum_{k=0}^{K} a_k r^{(2k+1)/2} \sin((2k+1)\theta/2) + O(r^{(2K+1)/2})$$

Due to the orthogonality of the sine functions on the interval $0 \leqslant \vartheta \leqslant \pi$ the left-hand side of (2) can be replaced by the expression

$$-\tfrac{1}{2}(2k+1)\,\pi a_k + \varepsilon(\delta) \tag{3}$$

where $\varepsilon \to 0$ as $\delta \to 0$. Since the function v vanishes on the arc $r = A$, and the right-hand side of (2) is independent of δ, Eq. (1) can be solved by means of (3) for the coefficient a_k:

$$\begin{aligned} a_k &= -(2/(2k+1)\pi) \int_{\Gamma_A} u v_r \, ds \\ &= (2/\pi)\, A^{-(2k+1)/2} \int_0^{\pi} u(A, \theta) \sin((2k+1)\theta/2)\, d\theta. \end{aligned} \tag{4}$$

An approximation u_h of the function $u(A, \vartheta)$ can then be obtained by some appropriate method (for the calculations reported on in this paper, finite-difference equations on a uniform square mesh of side h were used, together with the nine-point Laplacian and SOR). The a_k can then be computed using (4) with some appropriate quadrature rule and with the approximation u_h replacing u. For the calculations in the present paper a Romberg quadrature was used. Since this approximation is more likely to be accurate away from the singular point, A should be taken as large as possible (in the present case $A = 1$).

Let the results of these computations be called $\tilde{b}_k^1$, and then define the first-order approximations $\tilde{a}_k^1$ to the a_k by $\tilde{a}_k^1 = \tilde{b}_k^1$, $k = 0, 1, ..., K$.

The boundary values of the original problem are then modified by subtracting off the series

$$w^1 = \sum_{k=0}^{K} \tilde{a}_k^1 r^{(2k+1)/2} \sin((2k+1)\theta/2).$$

On the top of the rectangle, e.g., the boundary condition would now be $\partial u/\partial n = -\partial w^1/\partial n$.

The finite-difference process is then repeated for this new boundary value problem, and new coefficients $\tilde{b}_k^2$ are computed. The second-order approximation of a_k is then defined by $\tilde{a}_k^2 = \tilde{b}_k^2 + \tilde{b}_k^1$. In general the mth approximation is defined by $\tilde{a}_k^m = \tilde{a}_k^{m-1} + \tilde{b}_k^m$ and the series

$$w^m = \sum \tilde{a}_k^m r^{(2k+1)/2} \sin((2k+1)\vartheta/2) \tag{5}$$

is used to modify the boundary values.

The process is then continued until the $\tilde{b}_k^m$ are smaller in magnitude than some prescribed upper limit.

Finally, the numerical solution of the original boundary value problem is constructed in the following way. At any point P with polar coordinates (r, ϑ) one takes the most recent solution $u_h(P)$ of the last modified boundary value problem and adds to it the series (5).

3. Applications of the Method

a. *The Problem of Motz*

In this section we shall give some comparisons between the results of the present method and those of [3, 5, 6, and 11]. The calculation of the coefficients a_k was carried out for $K \leqslant 13$. This was an arbitrary choice, as the method introduced in this paper allows these coefficients to be calculated in a very simple manner: the finite difference scheme on a regular square grid of size h is run and the solution u_h is inserted into (4) to calculate the coefficients. In most of the present calculations K was taken equal to 8 and the mesh size h varied between $\frac{1}{2}$ and $\frac{1}{13}$. The radius A of the path of integration was generally taken equal to 1.0, though reasonable results were also obtained with $A = 0.5$. See Tables I–III for comparisons.

The question of convergence of the asymptotic expansion (1) is, in general, a difficult one. For the problem at hand, however, the matter is easily settled. According to Hadamard's theorem the radius of convergence of (1) is given by the reciprocal of the quantity $\limsup |a_k|^{1/k}$ and the latter is easily shown to be equal to A^{-1}. Thus the series converges absolutely and uniformly for $r < A$. A can clearly be taken equal to 1 but an improvement can be made by the following simple argument. By reflecting the function $u - 500$ across the top of the rectangle as an even function of y we get a continuation of $u - 500$ to the rectangle $-1 < x < 1$, $0 < y < 2$. This function can then be extended to the left by reflection as an odd function, and to the right as an even function. It is thus seen that the series for $u - 500$, and thus the series for u converges uniformly and absolutely for $r < 2$. Since the new extended solution obviously has a singularity at the point (0, 2), the radius of convergence of the series (1) is equal to 2.

TABLE I

Numerical Solution of the Problem of Motz—First Comparison

(x_i, y_i)	$(\frac{2}{7}, \frac{2}{7})$	$(0, \frac{2}{7})$	$(-\frac{2}{7}, \frac{2}{7})$	$(\frac{1}{28}, \frac{1}{28})$	$(0, \frac{1}{28})$	$(-\frac{1}{28}, \frac{1}{28})$	$(-\frac{1}{28}, 0)$	$(-\frac{3}{28}, 0)$
Li	78.4732	141.133	243.567	33.5478	53.1192	83.5686	76.315	
Thatcher	78.24	140.9	243.3	33.37	52.89	83.20	76.01	
W and P	78.56	141.6	243.8	33.59	53.19	83.67	76.41	
Wigley	78.559	141.560	243.812	33.592	53.186	83.671	76.408	134.447
XBA							76.41	134.45

TABLE II

Numerical Solution of the Problem of Motz—Second Comparison

(x_i, y_i)	$(0, \frac{1}{7})$	$(-\frac{1}{7}, 0)$	$(\frac{6}{7}, \frac{6}{7})$
Gregory *et al.*	103.78	156.40	90.83
W and P	103.77	156.48	91.34
Wigley	103.768	156.483	91.343

To give an indication of how accurate the series representation of the solution is, the coefficients $\tilde{a}_k$ were calculated for $k \leqslant K = 13$ with $h = \frac{1}{13}$. Iteration was carried out until $|\tilde{b}_1^m| < 10^{-6}$. Values of the solution were then approximated by the truncated series on the left side of the rectangle, at the points $(-1, j/10)$, for $j = 0, 1, ..., 10$. The true solution is of course equal to 500 at these points. The greatest error in the calculation occurred at $y = 0.8$ with a calculated value of 500.0036 and a relative error of 7.2×10^{-6}.

b. *The Cracked Beam of Fix et al.*

A similar study has been carried out for the problem of torsion of a cracked beam with a square cross section (see Fix *et al.* [2]). After symmetrizing, the problem is reduced to solving Poisson's equation in the rectangle with vertices $x = \pm\frac{1}{2}$, $y = 0$, $\frac{1}{2}$ and with boundary values as indicated in Fig. 2. In [2] the problem was solved by introducing the singular functions of the asymptotic expansion (1) to the finite element space of trial functions which used various splines of degrees one and three. To check for accuracy the authors also used quintic splines.

TABLE III

N	Wigley	XBA	Li	Thatcher	Symm
0	401.163	401.175	400.665	400.8	401.2
1	−87.655	−87.601	−87.7679	−88.0	−87.2
2	17.238	17.236	17.6683	17.3	
3	8.071	8.049	9.66311		
4	1.440	1.398	1.79988		
5	−0.331				
6	0.275				
7	0.087				
8	0.0336				
9	−0.0154				
10	0.0073				
11	0.0032				
12	0.0012				
13	−0.0005				

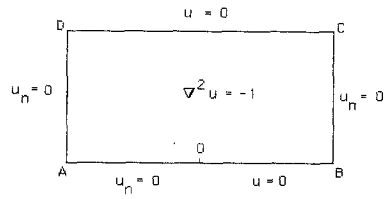

FIG. 2. The cracked square (symmetrized) of Fix *et al.*

In engineering problems of this kind a very important constant is the so-called stress intensity factor, which in the present case is the coefficient a_0. This constant gives a measure of "the amount of torsion the beam can endure before fracture occurs" [2].

The problem can be reduced to Laplace's equation by considering the function $v = u + y^2/2$, which is harmonic and has the same boundary values as u, except on the top of the rectangle where $v(x, \frac{1}{2}) = \frac{1}{8}$. The method applied earlier to the problem of Motz works equally well here. First, the asymptotic series for the function v has (by an argument almost identical to the one given earlier) a radius of convergence equal to one (observe that the rectangle here is half the size of the previous one). Our calculated results are given in Tables IV and V. It must be pointed out that the coefficients a_2 and a_3 given in Table V were *not* equal to zero, the values listed in the table having been merely rounded off.

4. APPROXIMATIONS OF KNOWN SOLUTIONS

Also considered were some problems in a domain with a slit. Let u be harmonic on the square with vertices $(\pm 1, \pm 1)$, with the exception of a slit along the positive x-axis (see Fig. 3). Let the function u vanish on the upper side of the slit, and let the normal derivative $\partial u/\partial n$ vanish on the lower side of the slit. On the outer boundary of the square let u be defined by $u = \Phi$, where Φ is given by

$$\Phi = \sum_{k=0}^{\infty} a_k r^{(2k+1)/4} \sin((2k+1)\theta/4). \tag{6}$$

TABLE IV

Numerical Solution of the Cracked Beam Problem

(x, y)	$(0, \frac{1}{24})$	$(\frac{11}{24}, \frac{1}{4})$	$(-\frac{11}{24}, \frac{1}{4})$
Fix *et al.*	0.027425	0.032877	0.070844
Wigley	0.027428	0.032878	0.070844

TABLE V

Values of the Coefficients (Cracked Beam Problem)

k	0	1	2	3	4	5
Fix *et al.*	0.1917					
Wigley	0.19112	0.11811	0.00000	0.00000	−0.01256	0.01905

The solution to the boundary value problem is, of course, $u \equiv \Phi$. Some test problems were considered by various judicious choices of the coefficients a_k. Convergence studies were carried out in both the l^∞ and l^2 norms.

First considered were problems in which all but a finite number of the a_k are zero, say $a_k = 0$ for $k > m$. Approximations were then computed by selecting a non-negative integer K and computing an approximation to the solution u and thence approximations of the coefficients a_k for $0 \leqslant k \leqslant K$. It was found that if K is taken $\geqslant m$ then convergence was fast and extremely accurate (*to within machine-epsilon*).

If, on the other hand, K is not taken large enough, the "missed" coefficients, i.e., a_k for $K < k \leqslant m$, spoil the calculations unless the coefficients themselves are small. Thus in the general case (6) it would seem appropriate to experiment with various values of K.

Additionally, two infinite series were tested by choosing the a_k in (6) in such a way that the series can be summed. The first example was gotten by taking $a_k = (-1)^k/(2k+1)!$, for which the series (6) becomes

$$\begin{aligned}\Phi &= \operatorname{Im}\{\sin z^{1/4}\}\\ &= \cos(r^{1/4}\cos \vartheta/4)\sinh(r^{1/4}\sin \vartheta/4)\end{aligned}$$

and which converges for $r < \infty$. As indicated in Table VI, with $h = \frac{1}{3}$ and $K = 7$ the maximum error between the computed solution and the true solution occurred at the point $(\frac{2}{3}, \frac{2}{3})$ and was of the order 1.6×10^{-5}. The largest error in the computation of the coefficients was 5.3×10^{-7}.

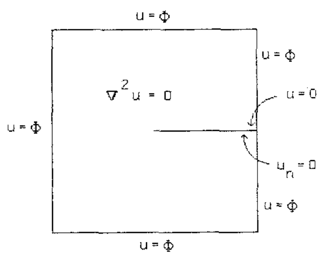

FIG. 3. The test problems.

TABLE VI

Comparisons with True Solutions

	First series		Second series	
	Orig. error	Final error	Orig. error	Final error
l^∞ norm	0.213	0.000016	0.173	0.0019
Location	$(\frac{1}{3}, 0)$	$(\frac{2}{3}, \frac{1}{3})$	$(\frac{1}{6}, 0)$	$(\frac{5}{6}, \frac{2}{3})$
l^2 norm	0.086	0.000004	0.046	0.00049
Max error of coeffs.	N/A	5.3×10^{-7}	N/A	7.1×10^{-4}

The second example of the infinite series (6) studied was formed by taking $a_k = 2^{-(2k+1)}$, which yields the geometric series

$$\Phi = \operatorname{Im} \frac{(z/2)^{1/4}}{1-(z/2)^{1/2}}$$

$$= \frac{(r/2)^{1/4} \sin(\theta/4)(1+(r/2)^{1/2})}{1-2(r/2)^{1/2} \cos(\theta/2)+r/2}$$

which converges only for $r < 2$. It will be noted in Table VI, in which "first" and "second" series refer to the first and second *infinite* series of this section, that convergence for the present case is nowhere near as good as for the first series. The difference seems to lie with the speeds of convergence of the two power series, the latter series converging quite slowly in comparison with the former. Nevertheless application of the method did improve convergence in both the l^∞ and l^2 norms by two orders of magnitude. Observe too that the maximum error for the second series occurs at the point $(\frac{5}{6}, \frac{1}{3})$, which is not near the corner.

TABLE VII

Comparisons with PLTMG

(x, y)	True value	PLMTG	rel error	Wigley	rel error
$(-0.1, -0.1)$	0.42596	0.41927	0.01571	0.42601	0.00011
$(-0.1, 0.1)$	0.41822	0.40880	0.02252	0.41820	0.00004
$(0.1, -0.1)$	0.40988	0.40400	0.01434	0.40989	0.00004
$(0.1, 0.1)$	0.21982	0.21727	0.01158	0.21987	0.00025
$(-0.8, -0.8)$	0.59002	0.58877	0.00213	0.59017	0.00025
$(-0.8, 0.8)$	0.85269	0.85062	0.00243	0.85249	0.00024
$(0.8, -0.8)$	0.50427	0.50311	0.00230	0.50422	0.00010
$(0.8, 0.8)$	1.68484	1.68450	0.00020	1.68499	0.00009

Finally, a comparison was made between the method given in this paper and results gotten by an application of PLTMG [1], an adaptive grid refinement code which uses approximations which are piecewise linear on finite elements. The second infinite series given above (the geometric one) was run using PLTMG. Results are compared in Table VII, the comparisons being made at the points x, $y = \pm 0.1$, ± 0.8 (eight points), these points having been chosen because half are "near" the corner and half are not. It should be observed that the methods used in this paper give equally good results at "near" and "far" points, whereas PLTMG is much more inaccurate near the corner, despite the fact that the grid is highly refined there.

5. Discussion

A method for subtracting off a singularity at a corner for Laplace's equation has been discussed. Though the boundary values in the examples listed above were of the mixed kind at the singularity, the method applies equally well to Dirichlet or Neumann conditions as well as to interior angles other than π or 2π, as can be seen by considering slight modifications of the function v of Section two.

The method is very easy to apply in that it can be accommodated by existing computer programs through only slight changes. It adapts to any method: boundary integral, finite difference, finite element, etc. If indeed high accuracy comes with a price, then the additional price to be paid here is low: the extra programming time is small, and extra computer time was not great; most of the results reported on in this paper involved three or four sweeps (one sweep meaning adjusting the boundary values, SOR, and quadrature), with a maximum of five. Finally, for the engineer who only wants a rough and quick approximation of a stress intensity factor, it is worth mentioning that one sweep may suffice, so that an existing program need only be coupled with one additional quadrature.

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